APPENDIX.

NOTES ON COURNOT'S MATHEMATICS.*

1. *p.* 34, *end.* $c_{3,2}$ is found by dividing the value of $c_{3,1}$ by the value of $c_{2,1}$, in accordance with the second line of equation (c), p. 33. The values of $c_{3,1}$ and $c_{2,1}$ were, of course, obtained by solving the two equations above them. (The mathematical reader will note that Cournot must have been unacquainted with determinants, which in his time were not widely used. Otherwise he would almost certainly have expressed the general solutions of (d) instead of restricting himself to the special case of three centres; also on pp. 110 and 115, Q and R would have been explained to be determinants.)

2. *p.* 36. I, the net money imported at (1), is the difference between the total debts to (1) from *all* sources; *i.e.*, not simply from (2) and the total debts from (1). Similarly for E. To verify equation (e), write it in full; *i.e.*, substitute the values of E and I. Two terms on each side will then drop out (remembering that $\gamma_{1,2}\,\gamma_{2,1} = 1$); and the result will be found identical with that obtained by adding together all equations (d) except the first two, remembering that $c_{2,1}$ is now $\gamma_{2,1}$ and that $c_{3,2}\,\gamma_{2,1} = c_{3,1}$, etc.

3. *p.* 53, *equation* (1). The value of p which will render $pF(p)$ a maximum is the root of the equation formed by putting the differential coefficient of $pF(p)$, namely, $F(p) + pF'(p)$, equal to zero.

4. *p.* 53, *first new paragraph.* The geometrical interpretation of maximizing pD is to maximize the rectangle On, since the area of this rectangle is the product of its base, Oq or p by its altitude, qn or D. It is a proposition of geometry that On is a maximum when n is so situated that $Oq = qt$. This equation, in fact, is a geometrical form of equation (1), which may be written $p = \frac{F(p)}{-F'(p)}$, where the left member is represented by Oq and the right member by qt (for $F(p)$ is nq and $F'(p)$, the slope of the curve at n, is $\frac{nq}{-qt}$, so that $\frac{F(p)}{-F'(p)} = \frac{nq}{\frac{nq}{qt}} = qt$.)

5. *p.* 54, § 25. To discriminate between the cases of maximum and minimum of $pF(p)$, resort is had to the *second* differential coefficient of $pF(p)$; *i.e.*, the differential coefficient of $F(p) + pF'(p)$ or $2F'(p) + pF''(p)$. According as this is negative or positive will the value of p belong to a maxi-

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mum or a minimum. This second differential coefficient may be transformed by substituting for p its value, $-\frac{F(p)}{F'(p)}$, obtained from (1), p. 53. The inequality thus obtained is next cleared of fractions by multiplying through by $F'(p)$, which, being a negative quantity, reverses the signs of inequality. The final result is that at the foot of p. 54. Examination of this result shows that the first term is necessarily positive, and that the second term, $-F(p)F''(p)$, will also be positive if $F''(p)$ is negative.

6. *p.* 56. Equation (1) gives $p = \frac{F(p)}{-F'(p)}$, which multiplied by $F(p)$ gives $pF(p) = \frac{[F(p)]^2}{-F'(p)}$.

7. *p.* 57, *equation* (2). To make the net receipts $pF(p) - \phi(D)$ a maximum, its differential coefficient must be zero; *i.e.*, $F(p) + pF'(p) - \frac{d\phi(D)}{dp} = 0$. Cournot's result (2) is the same, D being in place of $F(p)$, $\frac{dD}{dp}$ in place of $F'(p)$, and $\frac{d\phi(D)}{dD} \times \frac{dD}{dp}$ in place of $\frac{d\phi(D)}{dp}$. The form given on p. 61 (3) is nearer that here expressed.

8. *pp.* 61, 62. Supposing $\psi(p)$ to be replaced by $\psi(p) + u$, equation (3), namely,

$$F(p) + F'(p)\,[p - \psi(p)] = 0 \qquad (3)$$

becomes

$$F(p) + F'(p)\,[p - \psi(p) - u] = 0 \qquad (3)'$$

If the root of (3) is p_0, the root of (3)′ is called $p_0 + \delta$. (3) may then be written,

$$F(p_0) + F'(p_0)\,[p_0 - \psi(p_0)] = 0$$

and (3)′,

$$F(p_0 + \delta) + F'(p_0 + \delta)\,[p_0 + \delta - \psi(p_0 + \delta) - u] = 0.$$

Now, by Taylor's theorem, $F(p_0 + \delta) = F(p_0) + \delta F'(p_0) +$ terms involving squares and higher powers of δ, which may all be neglected, assuming δ sufficiently small and Taylor's theorem applicable. Substituting this value for $F(p_0 + \delta)$ and, in like manner, $F'(p_0) + \delta F''(p_0)$ for $F'(p_0 + \delta)$ and $\psi(p_0) + \delta\psi'(p_0)$ for $\psi(p_0 + \delta)$, we obtain another form of (3)′. If (3) be subtracted from this, the result is (4), after neglecting any terms which may remain involving increments of the second order, such as δ^2 or δu. The process, here exemplified, of deriving the relation between a small cause such as u and its effect δ, is so repeatedly employed by Cournot that the careful student will do well to master it once for all.

9. *p.* 63. The formula immediately preceding § 34 is obtained by substituting in the formula above it the value of $p_0 - \psi(p_0)$ derived from (3), namely,$-\frac{F(p_0)}{F'(p_0)}$, and then multiplying through by the *negative* quantity $F'(p_0)$, which reverses the signs of inequality.

10. *p.* 71. The value of $p' - p_0$ is derived just as was equation (4) on p. 62. In fact, (4), p. 62, and the present equation are identical except in form. The tax i here takes the place of the increase of cost, u; and $p' - p_0$, the increase of price, is the same magnitude as δ. The identity is seen by obtaining the value of δ from (4), p. 62, multiplying numerator and denomi-

nator by $F'(p_0)$, and substituting for $F'(p_0)\ [p_0 - \psi(p_0)]$ in the denominator its equal,— $F(p_0)$, as given by the first equation on p. 71.

11. *p* 72, *line* 2. *I.e.*, the loss is the difference between the net income at price p_0 and the net income at price p', which latter net income involves the deduction of the tax $iF(p')$.

12. *p*. 72, *line* 7. The left member, being the *maximum* value of the function two lines above, is necessarily greater than the right member, which is *another* value of that function.

13. *p*. 72. The third formula from the bottom is obtained by adding the two inequalities which follow.

14. *p*. 73, *first equation*. See note 3.

15. *p*. 74, *last equation*. See note 11.

16. *p*. 77, *line* 7. Of the two cases, the second begins on p. 78, line 3, *not* on p. 77, last paragraph, which relates only to a subdivision of case one.

17. *p*. 82, *line* 5, *ff*. To put $D_1 = 0$ *in equation* (1) means to ask the question, Under what circumstances would producer (1) find it profitable to make $D_1 = 0$; *i.e.*, to cease producing entirely? The answer is, When $f(D_2) = 0$. Since $D = D_1 + D_2 = D_2$, $f(D_2)$ is $f(D)$ or p (p. 80, line 7). Hence $p = 0$. In fact, it is evident *a priori* that producer (1) would cease producing only when D_2, the output of his rival, is large enough to make the price zero. To put $D_1 = 0$ *in equation* (2), on the other hand, is to ask, What would producer (2) do if producer (1) withdrew from the field? The answer is, He would then be a simple monopolist, and would maximize pD_2. To accomplish this, p cannot be zero. That is, in both cases D_2 represents the total output; but, in the first case, this output is large enough to reduce the price to zero, while in the second it is not. Hence D_2 in the first case exceeds D_2 in the second.

18. *p*. 82. Equation (3) is derived from the preceding equation by substituting p for $f(D)$ and $\frac{dp}{dD}$ (this being the same as $\frac{df(D)}{dD}$) for $f'(D)$, and then dividing through by $\frac{dp}{dD}$.

19. *p*. 83, § 45. Here x is put for p. y has no special enonomic significance. The manner in which the intersection of the two curves corresponds to the solution of equation (3) is that the x of the intersection is equal to the value of p which satisfies (3). The reason for this is that for the point of intersection the co-ordinates of the two curves are equal; and, since the y of one is equal to $2x$ and the y of the other to $-\frac{F(x)}{F'(x)}$, we have $2x = -\frac{F(x)}{F'(x)}$. Since this equation is obviously a form of (3), the x which satisfies it is equal to the p which satisfies (3).

20. *p*. 84, *top*. The condition that the curve must fulfil in order that the result given may follow is incompletely stated. It should have been added that the value of the function for $x = 0$ (as well as for $x > 0$) must be positive.

21. *p* 84. Equations (5) are simply the general case, for n producers, of equations (1) and (2) on p. 81 for two producers.

22. *p.* 85. Equations (6) are the conditions that the profit of each producer shall be a maximum. That profit is no longer given by the expression on p. 80, line 10, but by that expression less the cost of production, $\phi_1(D_1)$ for producer (1), or $\phi_2(D_2)$ for producer (2), etc. The differential coefficient of this new expression for the profit of each producer will give the equations (6).

23. *p.* 85, *three lines from bottom.* $\frac{dD}{dp}$ is negative because of the law of demand. An increment of the price p causes a decrement of the demand D.

24. *p.* 86. Equation (7) is obtained by dividing the previous equation by $f'(D)$ and replacing $f(D)$ by p.

25. *pp.* 87–89 are devoted to a difficult proof that the value of p derived from (8) is greater than that derived from (7). The statement in the first line of p. 87 is not necessarily true, though it does not affect the argument.

26. *p.* 87, *last two lines, and p.* 88, *top.* Equations (6), on p. 85, show that D_1 is a function of D. But D is a function of p. Hence D_1 is a function of p. This function Cournot calls $\psi_1(p)$ or $\psi_1(x)$. In like manner D_2 is a function of p, and is called $\psi_2(p)$ or $\psi_2(x)$, and so on.

27. *p.* 88. As in note 19, if we equate the right members of (a) and (b), we obtain the abscissa of the intersection. But the equation thus formed being identical with (7), p. 86, its root must also be the same as the root of (7).

28. *p.* 88, 10 *lines from bottom.* OP is the value of y in (b) when x is zero; *i.e.*, it is minus the long bracket, or $-\Sigma\psi_n(x)$. Similarly, OP' is the value of y in (b') when x is zero; *i.e.*,$-\psi(x)$. Since $\Sigma\psi_n(x) > \psi(x)$, OP is numerically greater than OP'.

29. *p.* 89, *line* 10. "Stop producing" means here "cease to extend production," not "go out of business."

30. *p.* 90. The first equation on this page is a form of any one of equations (6), p. 85. The subscript k stands for any one of the subscripts, 1, 2, etc. All that is necessary to see the identity between (6) and its new form is to replace $f(D)$ in (6) by p, $f'(D)$ by $\frac{dp}{dD}$, and divide through by the latter.

31. *p.* 93. The process for deriving the central equation on this page is identical with that explained in note 8. Namely, we write (3) thus: $\Omega\ (p_0) = F(p_0)$, and (4): $\Omega\ (p_0 + \delta - u) = F\ (p_0 + \delta)$, and subtract the first from the second after expanding the latter by Taylor's theorem.

32. *p.* 93. The two inequalities are evident, if we remember that u is *given* positive.

33. *p.* 96. Equation (5) may be derived as follows: The expression for profit is evidently $D_k p - \phi_k(D_k) - npD_k$; *i.e.*, it is the gross receipts less the cost of production and the amount of the tax. The condition of maximum requires as usual that the differential coefficient of this expression be zero. This differential coefficient is evidently the left member of (5) *plus* $D_k \frac{dp}{dD_k}$, which, however, may be neglected in accordance with the next note. Cournot, however, evidently had in mind a different method of

derivation. Otherwise his explanation regarding $\frac{dp}{dD_k}$ would have been placed earlier.

34. *p.* 96, *line* 15. $\frac{dp}{dD_k}$ is assumed small. That is, the effect on the price of increasing the product is assumed small per unit of product. This is not because D_k is small, although Cournot seems to say so.

35. *p.* 96. Equation (6) is derived with the aid of $p(1-n)-\phi'_k p(D_k)=0$, just as (3) on p. 91 was derived with the aid of $p-\phi'_k(D_k)=0$.

36. *p.* 97. Equation (7) should have "0" expressed as a lower limit in the integral, as in the case of the integrals below.

37. *p.* 98. The first equation is found, as usual, by the condition of maximum. We differentiate expression (9) on page 97, remembering that the differential coefficient of the integral is $\phi'_k(D_k)$.

38. *p.* 101. In equations (1) and (2) it should not be forgotten that the F' is not a differential coefficient with respect to p_1 or p_2, but with respect to $(m_1p_1+m_2p_2)$; *e.g.*, $F'(m_1p_1+m_2p_2)$ in (1) is $\frac{dF(m_1p_1+m_2p_2)}{d(m_1p_1+m_2p_2)}$, and not $\frac{dF(m_1p_1+m_2p_2)}{dp_1}$.

To derive equation (1) from the differential equation above it, $\frac{d(p_1D_1)}{dp_1}=0$, we observe that $\frac{d(p_1D_1)}{dp_1}=D_1+p_1\frac{dD_1}{dp_1}$, and substitute for D_1 its value as given in equation (*b*) on page 100 and for $\frac{dD_1}{dp_1}$ its value $m_1\frac{dF(m_1p_1+m_2p_2)}{d(m_1p_1+m_2p_2)} \times \frac{d(m_1p_1+m_2p_2)}{dp_1}$. The first of these two differential coefficients is $F'(m_1p_1+m_2p_2)$. The second is found by performing the differentiation indicated, treating p_2 as a constant, and is evidently simply m_1. With these substitutions and cancelling the common factor m_1, we reach equation (1).

39. *pp.* 101, 102. Equation (1) gives the value of p_1, which maximizes the profit of producer (1) *for an assumed value of* p_2. The question arises, What effect will a change in the assumed value of p_2 have on the resulting value of p_1? That is, in equation (1) what is the relation between increments of p_2 and p_1? In short, what is $\frac{dp_1}{dp_2}$? According as $\frac{dp_1}{dp_2} \gtrless 0$ will an increase of p_2 produce an increase or a decrease of p_1. Now, to obtain $\frac{dp_1}{dp_2}$, the rule is to take the differential coefficient of the left member of (1) with respect to p_2 treating p_1 as constant, and again with respect to p_1 treating p_2 as constant, and then divide the first differential coefficient by the second, and prefix the minus sign. Proceeding thus, we find the differentiation with respect to p_2 gives $m_2F'(p)+m_1p_1m_2F''(p)$ (using p for $m_1p_1+m_2p_2$ as per equation (*a*)), while that with respect to p_1 gives $2m_1F'(p)+m_1^2p_1F''(p)$. Divide the former expression by the latter, prefix the minus sign, and put the result in place of $\frac{dp_1}{dp_2}$ in the inequality $\frac{dp_1}{dp_2} \gtrless 0$. Then strike out the factor $-\frac{m_2}{m_1}$, and

(as this factor is negative) reverse the signs of inequality. We shall then have: $\frac{F'(p)+m_1p_1F''(p)}{2F'(p)+m_1p_1F''(p)} \lessgtr 0$. Substituting for m_1p_1 its value as derived from equation (1), namely, $-\frac{F(p)}{F'(p)}$, and clearing, we reach the required result as given at the top of p. 102.

40. *p.* 105. Equations (e_1) and (e_2) are derived by the same process as that explained in note 38.

41. *p.* 106. The first equation is derived from (*b*), p. 100.

42. *p.* 106, *first paragraph.* If $\phi'_1(D_1)$ is a constant, it signifies that the cost of each additional unit is the same. Cournot tacitly assumes, *besides* this, that the cost vanishes when the product vanishes, so that $\phi'_1(D_1)$ is the cost for *every* unit, and therefore, of course, the average cost for all units. That is, $\phi'_1(D_1) = \frac{\phi(D_1)}{D_1}$, which may also be proved analytically thus: We have given $\phi'_1(D_1) = \text{constant} = k$, or $\phi'_1(D_1)dD = kdD$. Integrating, we have $\phi_1(D_1) = kD_1 + C$. Since for $D_1 = 0$, $\phi(D_1) = 0$, we have $C = 0$. Hence $\frac{\phi(D_1)}{D_1} = k = \phi'_1(D_1)$.

It should be observed that the assumption $\phi_1(0) = 0$ is not often verified. Cournot seems to have ignored this fact in the present connection, though in another passage he brought it out distinctly (see p. 60, last four lines and ff.).

43. *p.* 106. Equation (f) is obtained by adding (e_1) and (e_2), p. 105, and applying (*a*), p. 100. The next two equations are found by adding and subtracting the two equations $m_1p_1 + m_2p_2 = p$, or equation (*a*), p. 100, and $m_1p_1 - m_2p_2 = \phi'_1(D_1) - \phi'_2(D_2)$, or the last equation on p. 105.

44. *p.* 106. Equation (f') is a form of (2), p. 57, or of (3), p. 61, it being remembered that $\phi(D)$ is now $\phi_1(D_1) + \phi_2(D_2)$, so that $\frac{d[\phi(D)]}{dD}$ is $\phi'_1(D_1)\frac{dD_1}{dD} + \phi'_2(D_2)\frac{dD_2}{dD}$, and remembering that equations (*b*), p. 100, show the values of $\frac{dD_1}{dD}$ and $\frac{dD_2}{dD}$ to be respectively m_1 and m_2.

45. *p.* 119. Equations (1) are same as (3), p. 91.

46. *p.* 119. Equation (2) states that the home supply plus the foreign supply equals the home plus foreign demand.

47. *p.* 120. To obtain (4), write (2) in the form $\Omega_a(p_a+\delta)+\Omega_b(p_b+(\delta+\varepsilon-\omega)) = F_a(p_a+\delta)+F_b(p_b+(\delta+\varepsilon-\omega))$, expand by Taylor's theorem, and subtract the sum of equations (1), p. 119. The next formula is derived in like manner.

48. *p.* 122, § 70. The first equation is simply (2), p. 119, the primes being omitted.

49. *p.* 122, § 70. In the second formula u is treated as an addition to ε, and δ as the resulting increment of p. The second formula is simply the first with $\varepsilon + u$ for ε and $p + \delta$ for p.

50. *p.* 122. As already stated, equations (6) are incorrect. If the preceding equation be expanded by Taylor's theorem and the penultimate equa-

tion subtracted therefrom, we obtain $\delta\Omega'_{a}(p)+(\delta+u)\Omega'_{b}(p)=\delta\ F'_{a}(p)+(\delta+u)\ F'_{b}(p)$. Solving this with respect to δ, we obtain what ought to be the first equation of (6). But, instead of the parenthesis $(\varepsilon+u)$, we shall find simply u. The true form of the second equation is now found by adding u to the result just obtained. This corrected equation will differ from that given in the book in not having the second term in the numerator, ε [. . .].

This serious error of Cournot in the derivation of (6) was evidently due to the habit, in applying Taylor's theorem, of dropping automatically the first term in each expansion instead of formally subtracting the previous equation. That is, Cournot, instead of subtracting the first equation in § 70 from the expansion of the second, virtually subtracted $\Omega_{a}(p)+\Omega_{b}(p)=F_{a}(p)+F_{b}(p)$, which is not a true equation.

51. *p.* 123. "1" should now read ". . . and numerically smaller than u; *i.e.*, the tax will always cause the commodity to fall in the exporting market by a quantity which will *always* be smaller than the tax." The reason for this inequality is that the fraction on the right member of the first equation of (6) is a proper fraction, for the denominator exceeds the numerator, containing, as it does, all the terms which the numerator contains, and others besides; and all the terms are positive. "2" should read "$\delta+u$ is *always* positive and less than u," a conclusion involved in the corrected form of the second equation of (6).

52. *p.* 123, *5th paragraph, line* 4. It is *not* necessary to consider two cases. In *either* case δ is positive and numerically less than u. Thus the article *must* rise on the exporting market, and, $\delta+u$ being negative, *must* fall on the importing market. δ has no relation to ε.

53. *p.* 125. Equation (7) holds true, even if ε is not a small quantity, like δ and u. If ε *is* small, its presence in (7) is superfluous. In deriving (7) from the preceding formula, Cournot evidently called, *e. g.*, $\Omega_{b}(p+\delta+\varepsilon-u)$ equal to $\Omega_{b}(p+\varepsilon)+(\delta-u)\Omega'_{b}(p+\varepsilon)$ (applying Taylor's theorem in such a way as to make $\delta-u$ the increment of $p+\varepsilon$); while, if ε is small, he could have called it equal to $\Omega_{b}(p)+(\delta+\varepsilon-u)\Omega'_{b}(p)$ (regarding $\delta+\varepsilon-u$ as an increment of p). This remark will serve to reconcile Cournot's result with what the student may obtain in attempting to follow him.